Particle Waves and Group Velocity

Particles with known energy

Consider a particle with mass m, traveling in the $+x$ direction and known velocity v_o and energy $E_\mathrm{o} = \frac{1}{2} m\mathrm{v}_\mathrm{o}$. The wavefunction that represents this particle is:

$$\Psi(x,t) = Ce^{jkx}e^{-j\omega t} \qquad [1]$$

where C is a constant and

$$k_\mathrm{o} = \frac{2\pi}{\lambda_\mathrm{o}} = \frac{1}{\hbar}\sqrt{2mE_\mathrm{o}} \qquad [2]$$

$$\omega_\mathrm{o} = 2\pi\nu_\mathrm{o} = \frac{E_\mathrm{o}}{\hbar} \qquad [3]$$

The envelope $\left|\Psi(x,t)\right|^2$ of this wavefunction is

$$\left|\Psi(x,t)\right|^2 = \left|C\right|^2 ,$$

which is a constant. This means that when a particle's energy is known exactly, it's position is completely unknown. This is consistent with the Heisenberg Uncertainty principle.

Even though the magnitude of this wave function is a constant with respect to both position and time, its phase is not. As with any type of wavefunction, the phase velocity v_p of this wavefunction is:

$$\mathrm{v}_p = \frac{\omega}{k} = \frac{E_\mathrm{o}/\hbar}{\frac{1}{\hbar}\sqrt{2mE_\mathrm{o}}} = \sqrt{E_\mathrm{o}/2m} = \sqrt{\mathrm{v}_\mathrm{o}^2/4} = \frac{\mathrm{v}_\mathrm{o}}{2} \qquad [4]$$

At first glance, this result seems wrong, since we started with the assumption that the particle is moving at velocity v_o. However, only the magnitude of a wavefunction contains measurable information, so there is no reason to believe that its phase velocity is the same as the particle's velocity.

Particles with uncertain energy

A more realistic situation is when there is at least some uncertainly about the particle's energy and momentum. For real situations, a particle's energy will be known to lie only within some band of uncertainly. This can be handled by assuming that the particle's wavefunction is the superposition of a range of constant-energy wavefunctions:

$$\Psi(x,t) = \sum_n C_n \left(e^{jk_n x} e^{-j\omega_n t} \right) \qquad [5]$$

Here, each value of k_n and ω_n correspond to energy E_n, and C_n is the probability that the particle has energy E_n.

Let's now consider the simplest possible case, there a particle is known to have one of two equally probably, closely-spaced energies (and corresponding velocities), given by,

$$\begin{aligned} E_+ &= E_o + \Delta E \\ E_- &= E_o - \Delta E \end{aligned} \qquad [6]$$

Here, $E_o = \frac{1}{2} m \mathrm{v}_o^2$ is the mean energy, where v_o is the mean velocity. The corresponding particle wavefunction for this particle is

$$\Psi(x.t) = Ce^{jk_+ x} e^{-j\omega_+ t} + Ce^{jk_- x} e^{-j\omega_- t} \qquad [7]$$

where,

$$\omega_\pm = \frac{E_o \pm \Delta E}{\hbar} = \omega_o \pm \Delta\omega \qquad [8]$$

and

$$k_\pm = \frac{1}{\hbar}\sqrt{2mE_\pm} = \sqrt{\frac{2m(\omega_o \pm \Delta\omega)}{\hbar}} \qquad [9]$$

However, if $\Delta\omega$ is small, we can use the binomial theorem to expand k_+ and k_- as:

$$k_\pm \approx \sqrt{\frac{2m\omega_o}{\hbar}} \left(1 \pm \frac{1}{2} \frac{\Delta\omega}{\omega_o} \right) = k_o \pm \Delta k \qquad [10]$$

where

$$\Delta k = \Delta\omega \sqrt{\frac{m}{2\hbar\omega_o}} = \Delta\omega \sqrt{\frac{m}{2E_o}} = \frac{\Delta\omega}{\mathrm{v}_o} \qquad [11]$$

This allows us to write the wavefunction as:

$$\Psi(x.t) = Ce^{jk_o x} e^{-j\omega_o t} \left[e^{j\Delta k x} e^{-j\Delta\omega t} + e^{-j\Delta k x} e^{j\Delta\omega t} \right] \qquad [12]$$

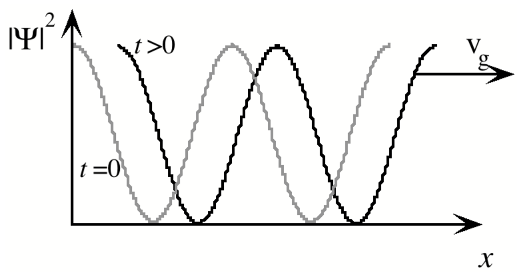
Using Euler's identity, the wavefunction becomes

$$\Psi(x,t) = 2Ce^{jk_o x}e^{-j\omega_o t}\cos(\Delta\omega t - \Delta k x) \qquad [13]$$

The envelope of this wavefunction is the density function of the wave packet:

$$|\Psi(x,t)|^2 = 4|C|^2\cos^2(\Delta\omega t - \Delta k x) \qquad [14]$$

Unlike the constant envelope for a particle with a uniquely known energy, *this* envelope is clearly a function of both time and position, as shown in the figure below.



As can be seen from this figure, this particle is most likely to be found at positions where $\cos^2(\Delta\omega t - \Delta k x)$ is the largest, and the regions where that occurs move to the right with increasing time with a constant velocity. This velocity is called the *group velocity*, since it's the velocity of the envelope of a group (in this case, 2) of waves traveling together. The velocity of the envelope function given by equation 14 is

$$\mathrm{v}_g = \frac{\Delta\omega}{\Delta k} \quad , \qquad [15]$$

which, using equation 11 yields:

$$\mathrm{v}_g = \mathrm{v}_o$$

This agrees with our starting assumption the particle has a mean velocity of v_o.

Even though we derived the above expression for group velocity in terms of a two-energy state particle, equation 15 is valid for particles with continuous uncertainties of energy. This means that the velocity of a particle is controlled by how its frequency varies with its wavenumber. In the limit as $\Delta E \to 0$, [15] can be expressed as

$$\mathrm{v}_g = \frac{\partial\omega}{\partial k} = \left[\frac{\partial k}{\partial\omega}\right]^{-1} \qquad [16]$$

This formula applies to waves of all kinds, including both matter and light wavefunctions.

For electromagnetic waves, ω and k in a vacuum are related by:

$$k = \omega\sqrt{\mu\varepsilon} \quad \text{(electromagnetic waves)} \qquad [17]$$

where μ and ε are the permeability and permittivity of the medium, respectively. Hence, the group velocity of an electromagnetic wave is

$$\mathrm{v}_g = \frac{\partial\omega}{\partial k} = \left[\frac{\partial k}{\partial\omega}\right]^{-1} = \frac{1}{\sqrt{\mu\varepsilon} + \omega\dfrac{\partial}{\partial\omega}\sqrt{\mu\varepsilon}} \quad \text{(electromagnetic waves)} \qquad [18]$$

If μ and ε are independent of frequency, then $\mathrm{v}_g = 1/\sqrt{\mu\varepsilon}$, which means that the group velocity equals the phase velocity. Such media are called *nondispersive media*.

For deBroglie (mass) waves, the particle frequency is a linear function of the particle energy E, so it is typical to write the group velocity in the following form:

$$\mathrm{v}_g = \frac{\partial\omega}{\partial k} = \frac{\partial\omega}{\partial E}\frac{\partial E}{\partial k} = \frac{1}{\hbar}\frac{\partial E}{\partial k} \quad \text{(deBroglie waves)} \qquad [19]$$

Hence, the velocity of a particle is governed by how it's energy changes with respect to its wavenumber.

For a free particle with velocity v_o, $E = \frac{1}{2}m\mathrm{v}_\mathrm{o}^2$ and $k = \frac{2\pi}{\lambda} = \frac{m\mathrm{v}_\mathrm{o}}{\hbar} = \frac{1}{\hbar}\sqrt{2mE}$, so $E = \frac{\hbar^2 k^2}{2m}$. From [19], we obtain

$$\mathrm{v}_g = \frac{1}{\hbar}\frac{\partial E}{\partial k} = \frac{1}{\hbar}\frac{\partial}{\partial k}\left[\frac{\hbar^2 k^2}{2m}\right] = \frac{1}{\hbar}\frac{\hbar^2 k}{m} = \frac{1}{\hbar}\frac{\hbar^2}{m}\frac{m\mathrm{v}_\mathrm{o}}{\hbar} = \mathrm{v}_\mathrm{o}$$

which is the expected result.

Equation [19] is valid even when the particle is in a force field, i.e., regions where the potential function $V(x)$ varies with position. In that case, the relationship between E and k will *not* be $k = \frac{1}{\hbar}\sqrt{2mE}$ as it is for a particle that is traveling without the influence of external forces.